A note on interference between independent bosonic fields under U(1) superselection rule

KAWAKUBO Toru
http://kwkbtr.info

September 24, 2010

Abstract

When describing quantum interference between independent bosonic fields under U(1) superselection rule (SSR), it is common to employ states which violate the SSR. I discuss validity of the use of such states using correlation functions of measured values. It is shown that such states are indeed valid under some circumstances.

1 Introduction

Interference is one of the most impressive phenomena in quantum theory. In particular, interference in many-body systems like superconductors or Bose-Einstein condensates (BECs) of atoms is a macroscopic quantum effect and has been attracting much interest for years. Interference of light beams from lasers also is a result of quantum theory when the light fields are treated as quantum fields. While the first observation of interference fringes between two independent laser fields was many years ago [1], those between two independent atomic BECs had not been observed until recently [2].

When describing such quantum interference, it is common to employ a state like a coherent state [3], which is a superposition of eigenstates of particle number, e.g., number of charges in a superconductor, number of atoms in an atomic BEC and number of photons in a laser field. Since such a state has a nonvanishing mean field value, it is possible to describe interference in the same manner as in the case of classical fields.

On the other hand, many authors have questioned validity of taking these states, which violate superselection rules. A superselection rule (SSR) is a rule forbidding some superposition states. An SSR is usually associated with a certain symmetry of the system [4]. For example, when the system has a U(1) symmetry, any state $\hat{\rho}$ of the system must be invariant under the U(1) action $\hat{U}(\varphi) = \exp(i\hat{n}\varphi)$, i.e., $\hat{U}(\varphi)\hat{\rho}\hat{U}^\dagger(\varphi) = \hat{\rho}$. Thus the state $\hat{\rho}$ and the particle number $\hat{n}$ must commute, which means $\hat{\rho}$ has no superposition of states with different particle numbers. The system of electrons and the system of atoms are both subject to the U(1)-SSR and superpositions of states with different charges or atoms are not permitted. The system of photons also obeys the U(1)-SSR, due to conservation of energy in quantum optics [4, 5].

To justify the use of coherent states, it is usually claimed that the symmetry is "spontaneously broken." However, in general, there is no mechanism which breaks the symmetry in a real physical system, especially when the whole system, consisting of the main system and its environment, is treated as a quantum system. Therefore it should be explained why two indepedent quantum fields exhibit interference, even though they have no superpositions and their mean field values vanish.

Many works have been devoted to this subject. (See, e.g., [4–14] and references therein.) It has been emphasized in some of them that back action of measurement to the system must be described explicitly and that entanglements between subsystems play a crucial role [5]. It has also been suggested that whether the description of the system with coherent states is valid or not depends on the choice of reference frames [4]. Although some of these statements capture the essence, only a few quantitative and analytical theories which justify the use of coherent states exist. In this note, I show that taking coherent states is indeed valid by comparing observable quantities in two cases where coherent states or corresponding mixed states are employed to describe the system. The result here is quantitative and analytical one, and it provides a firm reason for the use of coherent states.

2 Coherent states violate U(1)-SSR

Throughout this note, I focus on the case of the U(1)-SSR. Under the U(1)-SSR, the particle number $\hat{n}$ is the superselected quantity. A general pure state of the system

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle, \qquad \sum_{n=0}^{\infty} |c_n|^2 = 1, \tag{1}$$

where $|n\rangle$ is some eigenstate of $\hat{n}$ with the eigenvalue n, violates the SSR unless the coeffeicients c_n are zero for all n other than some specific number m, i.e., $c_n = \delta_{nm}$. In other words, only pure states permitted under the SSR are eigenstates of $\hat{n}$.

On the other hand, the pure state (1) has conventionally been utilized as a "physical" description of the system when discussing interference. Consider observing interference between two fields $\hat{a}$ and $\hat{b}$. As we observe the number of particles in the superposed fields $\hat{a}+\hat{b}$, the interference term is $\hat{a}^\dagger \hat{b}$ (and its Hermitian conjugate). Assuming that the two fields are initially independent, the expectation value of the interference term factorizes, i.e., $\langle \hat{a}^\dagger \hat{b}\rangle = \langle \hat{a}^\dagger\rangle\langle \hat{b}\rangle$. If either of the initial states of the fields does not have superposition of different number states, the expectation value vanishes. Therefore states violating the SSR is necessary for a nonvanishing expectation value of the interference term.

However, as long as the SSR exists, there should be a way to describe the interference with states obeying the SSR. There are two keys to solve this puzzle. First, in an interference experiment, one measures *multiple* quantities on the *single* system. For example, if one wants to observe spatial interference in a system of atomic BECs, one measures values of atomic density at *many* spatial positions. In such a situation, one should take into account effect of back action of the measurements on the system [5, 6, 8, 10] or, alternatively, deal with a joint probability of the measurements. The second point is that it is not the expectation value that one measures in a single experiment. The vanishing expectation value does not imply that one cannot observe interference. Rather, the interference should be seen in the *multiple* observed values from a *single* experiment. If the position of the interference fringes varies from experiment to experiment and one takes an average of the fringes over experiments, they would be washed out, which is consistent with the vanishing expectation value [7, 12].

With these points in mind, I discuss in the rest of this note how one can predict the interference quantitatively. We consider correlation functions of measured values in experiments, which characterize probability distributions of them.

3 Correlation functions of particle numbers

Let us consider a system of two bosonic fields $\hat{A}$ and $\hat{B}$, which consists of identical bosons in two orthonormal modes ψ and ϕ. As usual, $\hat{A}$ and $\hat{B}$ satisfy the commutation relations of bosons:

$$[\hat{A}, \hat{A}^\dagger] = [\hat{B}, \hat{B}^\dagger] = 1, \quad [\hat{A}, \hat{B}] = [\hat{A}, \hat{B}^\dagger] = 0. \tag{2}$$

For simplicity, assume there is no interaction between the particles. In experiments we count particle number in multiple distinct modes f_l $(l = 1, \ldots, M)$. Note that f_l are assumed to be orthonormal since they are mode functions of distinct detectors. Annihilation operators $\hat{a}_l$ and $\hat{b}_l$ which correspond to these modes are defined by expansion of $\hat{A}$ and $\hat{B}$ as

$$\hat{A}\psi = \sum_{l=1}^{M} \hat{a}_l f_l \quad \text{and} \quad \hat{B}\phi = \sum_{l=1}^{M} \hat{b}_l f_l. \tag{3}$$

$\hat{a}_l$ and $\hat{b}_l$ are again subject to the commutation relations:

$$[\hat{a}_l, \hat{a}_{l'}^\dagger] = [\hat{b}_l, \hat{b}_{l'}^\dagger] = \delta_{ll'}, \tag{4}$$

$$[\hat{a}_l, \hat{a}_{l'}] = [\hat{b}_l, \hat{b}_{l'}] = [\hat{a}_l, \hat{b}_{l'}] = [\hat{a}_l, \hat{b}_{l'}^\dagger] = 0. \tag{5}$$

Further assume that all modes orthogonal to ψ and ϕ contain no particles. Thus all we measure are particle numbers $\hat{n}_l$ in superposed fields $\hat{a}_l + \hat{b}_l$:

$$\hat{n}_l = \left(\hat{a}_l^\dagger + \hat{b}_l^\dagger\right)\left(\hat{a}_l + \hat{b}_l\right). \tag{6}$$

Probability distribution of the measurement value $\{\hat{n}_l\}$ is completely characterized by correlation functions of all orders $\langle \hat{n}_1^{t_1} \cdots \hat{n}_M^{t_M} \rangle$. Since any product of annihilation and creation operators can be rearranged as a sum of normal-ordered products of equal or lower order, we only consider expectation values of the normal-ordered products

$$\hat{G} = \hat{G}_a \hat{G}_b \tag{7}$$

$$= \left[\prod_{l=1}^{M} (\hat{a}_l^\dagger)^{s_l'} \hat{a}_l^{s_l}\right] \left[\prod_{l=1}^{M} (\hat{b}_l^\dagger)^{t_l - s_l'} \hat{b}_l^{t_l - s_l}\right]. \tag{8}$$

We assume that the U(1)-SSR is present in the system and that states of the two fields $\hat{A}$ and $\hat{B}$ are independently prepared. The state of the system is then written as a tensor product $\hat{\rho}\otimes\hat{\sigma}$, where $\hat{\rho}$ and $\hat{\sigma}$ both obey the U(1)-SSR. We define the expectation value of any operator $\hat{O}$ with respect to this tensor product state as

$$\langle\hat{O}\rangle_{\mathrm{SSR}} = \mathrm{Tr}[(\hat{\rho}\otimes\hat{\sigma})\hat{O}]. \tag{9}$$

It is obvious that $\langle\hat{G}\rangle_{\mathrm{SSR}}$ factorizes as

$$\langle\hat{G}\rangle_{\mathrm{SSR}} = \langle\hat{G_a}\rangle_{\mathrm{SSR}}\langle\hat{G_b}\rangle_{\mathrm{SSR}}. \tag{10}$$

Since $\hat{\rho}$ and $\hat{\sigma}$ are subject to U(1)-SSR, this expectation vanishes unless $\sum_l s_l = \sum_l s'_l$.

We consider another state assignment, which violates the SSR. Let $\hat{\rho}_l$ and $\hat{\sigma}_l$ be states of fields $\hat{a}_l$ and $\hat{b}_l$ which might *not* obey the SSR. For a *single* run of the experiment, the state of the system might be seen as a tensor product

$$\hat{\rho}_1\otimes\cdots\otimes\hat{\rho}_M\otimes\hat{\sigma}_1\otimes\cdots\otimes\hat{\sigma}_M. \tag{11}$$

The expectation value of any operator $\hat{O}$ with respect to this states is defined as

$$\langle\hat{O}\rangle_{\mathrm{coh}} = \mathrm{Tr}[(\hat{\rho}_1\otimes\cdots\otimes\hat{\rho}_M\otimes\hat{\sigma}_1\otimes\cdots\otimes\hat{\sigma}_M)\hat{O}]. \tag{12}$$

During *several* runs, relative phase between $\hat{A}$ and $\hat{B}$ can take random values. Therefore, the overall density operator is written as a mixture

$$\int\frac{d\xi}{2\pi}\,e^{i\Delta\hat{N}\xi}\hat{\rho}_1\otimes\cdots\otimes\hat{\rho}_M\otimes\hat{\sigma}_1\otimes\cdots\otimes\hat{\sigma}_M e^{-i\Delta\hat{N}\xi}, \tag{13}$$

where $\Delta\hat{N} = \sum_l(\hat{a}_l^\dagger\hat{a}_l - \hat{b}_l^\dagger\hat{b}_l)$ is the particle number difference between $\hat{A}$ and $\hat{B}$. The expectation value of $\hat{G}$ with respect to this mixture is calculated as

$$\delta_{s,s'}\prod_{l=1}^{M}\left\langle(\hat{a}_l^\dagger)^{s'_l}\hat{a}_l^{s_l}\right\rangle_{\mathrm{coh}}\left\langle(\hat{b}_l^\dagger)^{t_l-s'l}\hat{b}_l^{t_l-s_l}\right\rangle_{\mathrm{coh}}, \tag{14}$$

where $s=\sum_l s_l$ and $s'=\sum_l s'_l$. Comparing Eqs. (10) and (14), it turns out that if

$$\left\langle\prod_{l=1}^{M}(\hat{a}_l^\dagger)^{s'_l}\hat{a}_l^{s_l}\right\rangle_{\mathrm{SSR}} = \delta_{s,s'}\prod_{l=1}^{M}\left\langle(\hat{a}_l^\dagger)^{s'_l}\hat{a}_l^{s_l}\right\rangle_{\mathrm{coh}} \tag{15}$$

(and the equivalent expression about $\hat{G}_b$) holds for order s up to t, the correlation functions of particle numbers of order up to t have the same values for the two state assignment $\hat{\rho}\otimes\hat{\sigma}$ and Eq. (11).

Since there are no particles in modes orthogonal to ψ and ϕ, the state of field $\hat{A}$ which obeys the SSR can be written as

$$\hat{\rho} = \sum_{N=0}^{\infty} p_N|N\rangle\langle N|, \tag{16}$$

where $|N\rangle = (\hat{A}^\dagger)^N/\sqrt{N!}|0\rangle$ is an eigenstate of the particle number $\hat{A}^\dagger\hat{A}$ with the eigenvalue N. From Eq. (3), $\hat{A}$ can be written in terms of $\hat{a}_l$ as

$$\hat{A} = \sum_{l=1}^{M} c_l\hat{a}_l = \sum_{l=1}^{M}(\psi, f_l)\hat{a}_l, \tag{17}$$

where $(\cdot,\cdot)$ denotes the inner product of mode functions. $|N\rangle$ can be rewritten in representation with particle numbers in detector modes as

$$|N\rangle = \sqrt{N!} \sum_{\substack{\{n_l\} \\ \sum_l n_l = N}} \prod_{l=1}^{M} \frac{(c_l^*)^{n_l}}{\sqrt{n_l!}} |n_1, \ldots, n_M\rangle. \tag{18}$$

The expectation value of $\hat{G_a}$ can be evaluated as

$$\langle N|\hat{G}_a|N\rangle = \delta_{s,s'} \frac{N!}{(N-s)!} \prod_{l=1}^{M} c_l^{s'_l} (c_l^*)^{s_l}. \tag{19}$$

On the other hand, if we take $\hat{\rho}_l = |\alpha_l\rangle_l\langle\alpha_l|$, $\langle\hat{G}_a\rangle_{\mathrm{coh}}$ can be evaluated as

$$\langle\hat{G}_a\rangle_{\mathrm{coh}} = \prod_{l=1}^{M} (\alpha_l^*)^{s_l} \alpha_l^{s'_l}, \tag{20}$$

where $|\alpha_l\rangle_l$ is a coherent state

$$|\alpha_l\rangle_l = e^{-|\alpha_l|^2/2} \sum_{n_l=0}^{\infty} \frac{\alpha_l^{n_l}}{\sqrt{n_l!}} |n_l\rangle_l. \tag{21}$$

Fock state In the case of Fock state $\hat{\rho} = |N\rangle\langle N|$, we can choose $\alpha_l = c_l^* \sqrt{N}$. Note that a product of the coherent states becomes the coherent state $|\sqrt{N}\rangle$ of the entire field $\hat{A}$. Then

$$\delta_{s,s'} \langle\hat{G}_a\rangle_{\mathrm{coh}} = \delta_{s,s'} N^s \prod_{l=1}^{M} c_l^{s'_l} (c_l^*)^{s_l} \tag{22}$$

holds. Correlation functions with order s typically have values of order $O(N^s)$. Difference between $\langle\hat{G}_a\rangle_{\mathrm{SSR}} = \langle N|\hat{G}_a|N\rangle$ and $\langle\hat{G}_a\rangle_{\mathrm{coh}}$ is

$$\delta_{s,s'} \frac{\langle\hat{G}_a\rangle_{\mathrm{SSR}} - \langle\hat{G}_a\rangle_{\mathrm{coh}}}{N^s} \sim O\left(\frac{1}{N}\right), \tag{23}$$

provided $s \ll N$. Therefore, with large particle number N, The Fock state description $|N\rangle$ and the coherent state description $|\sqrt{N}\rangle$ are practically equivalent.

Poissonian mixed state With a Poissonian mixture, i.e.,

$$p_N = e^{-\bar{N}} \frac{\bar{N}^N}{N!}, \tag{24}$$

where $\bar{N}$ is the average particle number, we choose $\alpha_l = c_l^* \sqrt{\bar{N}}$. In this case,

$$\langle\hat{G}_a\rangle_{\mathrm{SSR}} = \delta_{s,s'} \langle\hat{G}_a\rangle_{\mathrm{coh}}. \tag{25}$$

Thus, the Poissonian mixture of Fock states and the coherent state $|\sqrt{\bar{N}}\rangle$ are completely equivalent. This fact is also easily seen from decompositions of $\hat{\rho}$:

$$\hat{\rho} = e^{-\bar{N}} \sum_{N=0}^{\infty} \frac{\bar{N}^N}{N!} |N\rangle\langle N| \tag{26}$$

$$= \int \frac{d\varphi}{2\pi} \left|\sqrt{\bar{N}} e^{i\varphi}\right\rangle\!\left\langle\sqrt{\bar{N}} e^{i\varphi}\right|. \tag{27}$$

Thermal state We take the relevant mode ψ as an excited one-particle state with energy $\epsilon > 0$ and chemical potential μ. The particle number distribution of the thermal state under the inverse temperature β is written as

$$p_N = [1 - e^{-\beta(\epsilon-\mu)}]e^{-\beta(\epsilon-\mu)N} = \frac{1}{1+\overline{N}}\left(\frac{\overline{N}}{1+\overline{N}}\right)^N, \tag{28}$$

where $\overline{N} = [e^{\beta(\epsilon-\mu)} - 1]^{-1}$ is the average particle number. With this thermal state, the expectation value is

$$\langle \hat{G}_a \rangle_{\mathrm{SSR}} = \delta_{s,s'}\, s!\overline{N}^s \prod_{l=1}^{M} c_l^{s'_l} (c_l^*)^{s_l}. \tag{29}$$

Taking the coherent state amplitude as $\alpha_l = c_l^* \sqrt{\overline{N}}$, The difference between the two expectation values is

$$\delta_{s,s'} \frac{\langle \hat{G}_a \rangle_{\mathrm{SSR}} - \langle \hat{G}_a \rangle_{\mathrm{coh}}}{\overline{N}^s} \sim O(1). \tag{30}$$

We can thus conclude that for an excited one-particle state, as expected, the thermal state and corresponding coherent state assignment are significantly different.

The above analyses of single mode systems seem to work well. However, many realistic thermal systems like atomic BECs would require multi-mode analyses, since such one consists of many one-particle states distributed by a characteristic density of states and all such thermally populated states would affect the interference visibility. It can still be said that, intuitively, below the critical temperature of the BEC, the coherent state assignment to the ground mode would be valid, since the ground mode can be well approximated by the Poissonian mixture with a macroscopic average particle number whereas excited modes contain much less particles and the contribution of them would be negligible.

References

[1] G. Magyar and L. Mandel, "Interference Fringes Produced by Superposition of Two Independent Maser Light Beams," Nature **198**, 255–256 (1963)

[2] M. R. Andrews, C. G. Townsend, H.-J. Miesner, D. S. Durfee, D. M. Kurn, and W. Ketterle, "Observation of Interference Between Two Bose Condensates," Science **275**, 637–641 (1997)

[3] Roy Glauber, "Coherent and Incoherent States of the Radiation Field," Physical Review **131**, 2766–2788 (1963)

[4] Stephen Bartlett, Terry Rudolph, and Robert Spekkens, "Reference frames, superselection rules, and quantum information," Reviews of Modern Physics **79**, 555–609 (2007), arXiv:quant-ph/0610030

[5] Barry C. Sanders, Stephen D. Bartlett, and Peter L. Knight, "Photon-number superselection and the entangled coherent-state representation," Physical Review A **68**, 042329 (2003), arXiv:quant-ph/0306076

[6] Juha Javanainen and Sung Yoo, "Quantum Phase of a Bose-Einstein Condensate with an Arbitrary Number of Atoms," Physical Review Letters **76**, 161–164 (1996)

[7] M. Naraschewski, H. Wallis, A. Schenzle, J. Cirac, and P. Zoller, "Interference of Bose condensates," Physical Review A **54**, 2185–2196 (1996), arXiv:atom-ph/9606005

[8] Yvan Castin and Jean Dalibard, "Relative phase of two Bose-Einstein condensates," Physical Review A **55**, 4330–4337 (1997)

[9] Akira Shimizu and Takayuki Miyadera, "Charge superselection rule does not rule out pure states of subsystems to be coherent superpositions of states with different charges," (2001), arXiv:cond-mat/0102429

[10] Hugo Cable, Peter Knight, and Terry Rudolph, "Measurement-induced localization of relative degrees of freedom," Physical Review A **71**, 042107 (2005), arXiv:quant-ph/0411167

[11] Erich J. Mueller, Tin-Lun Ho, Masahito Ueda, and Gordon Baym, "Fragmentation of Bose-Einstein condensates," Physical Review A **74**, 033612 (2006), arXiv:cond-mat/0605711

[12] Anthony J. Leggett, *Quantum liquids: Bose condensation and Cooper pairing in condensed-matter systems* (Oxford University Press, New York, 2006) `http://books.google.com/books?id=a6c-JpbF85IC`

[13] G. Paraoanu, "Phase coherence and fragmentation in weakly interacting bosonic gases," Physical Review A **77**, 041605 (2008), arXiv:0804.1836

[14] Toru Kawakubo and Katsuji Yamamoto, "Optical homodyne detection in view of the joint probability distribution," Physical Review A **82**, 032102 (2010), `http://kwkbtr.info/pdf/PhysRevA.82.032102.pdf`