Interference between independently prepared Bose particles

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Introduction: interference in quantum physics

Bose-Einstein condensation

Interference experiment of independent BECs

Measurements induce interference & relative phase localization

Correlation function analysis

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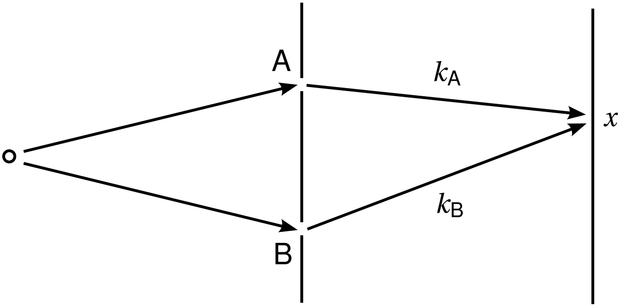
Interference of single particles

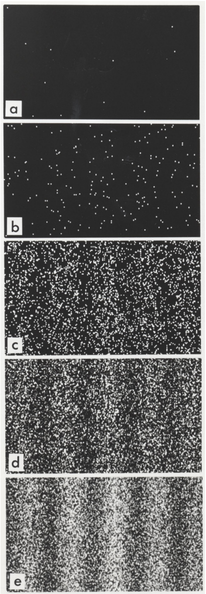


- Wave function $\psi(x) \propto e^{ik_A \cdot x} + e^{ik_B \cdot x}$
- Particle density $\rho(x) = P(x) \propto 1 + \cos \underline{(k_B - k_A) \cdot x}$

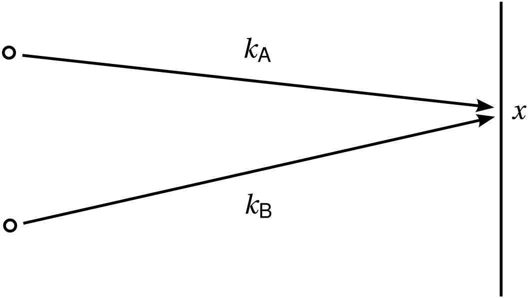
relative phase between paths A and B

Interference of single particles

- Of course, *one particle* does not exhibit interference.
- Each particle *independently* obeys *a single distribution.*
- *Interference pattern = particle density*



Interference of two Bose particles?



- Wave function $\psi(x_1, x_2) \propto e^{ik_\mathrm{A}\cdot x_1} e^{ik_\mathrm{B}\cdot x_2} + e^{ik_\mathrm{A}\cdot x_1} e^{ik_\mathrm{B}\cdot x_1}$
- Particle density

$$\rho(x) = 2\cdot|\psi(x,x)|^2 + 1\cdot\left[\int dx'\,|\psi(x,x')|^2 - |\psi(x,x)|^2\right] = \textbf{const.}$$

Dirac said

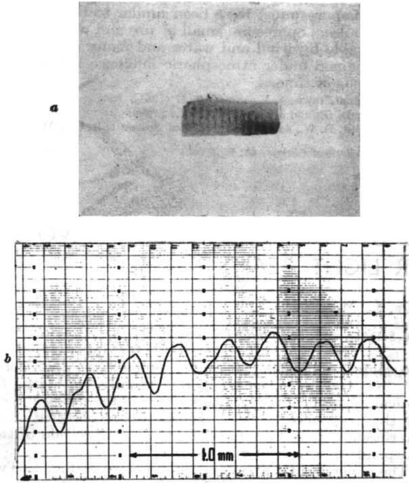
“Each photon then interferes only with itself. Interference between two different photons never occurs.”

Dirac said

"Each photon then interferes only with itself. Interference between two different photons never occurs."

Is this true?

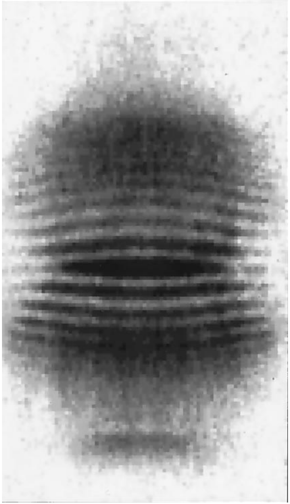
Interference between independent laser beams



Magyar and Mandel

Nature **198**, 255 (1963)

Interference between two independent BECs



Andrews *et al.*

Science **275**, 637 (1997)

Question

How can we understand the interference between *independent* Bose particles?

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Bose particles system

- Expansion of the field operator (Schrödinger picture)

$$\hat{\Psi}(\boldsymbol{x}) = \sum_k \hat{a}_k(t)\phi_k(\boldsymbol{x}, t)$$

$$\hat{a}_k(t) \equiv \int d^3x\, \phi_k^*(\boldsymbol{x}, t)\hat{\Psi}(\boldsymbol{x}) \quad \text{anihilation operator}$$

- With fixed t, “One-particle states” $\{\phi_k\}$ are orthonormal

$$\int d^3x\, \phi_k^*(\boldsymbol{x}, t)\phi_{k'}(\boldsymbol{x}, t) = \delta_{kk'}$$

and complete

$$\sum_k \phi_k(\boldsymbol{x}, t)\phi_k^*(\boldsymbol{x}', t) = \delta(\boldsymbol{x}' - \boldsymbol{x}).$$

Bose particles system

- Particle number

$$\hat{N} \equiv \int d^3x\, \hat{\Psi}^\dagger(\boldsymbol{x})\hat{\Psi}(\boldsymbol{x}) = \sum_k \hat{a}_k^\dagger(t)\hat{a}_k(t) \equiv \sum_k \hat{n}_k(t)$$

- Creation and anihilation operators $\hat{a}_k^\dagger(t)$ and $\hat{a}_k(t)$ satisfy

$$\left[\hat{a}_k(t), \hat{a}_k^\dagger(t)\right] = \delta_{kk'},$$

$$\hat{a}_k|\{n_k\}\rangle = \sqrt{n_k}|\ldots, n_k - 1, \ldots\rangle, \quad \hat{a}_k^\dagger|\{n_k\}\rangle = \sqrt{n_k + 1}|\ldots, n_k + 1, \ldots\rangle.$$

- Number basis

$$|\{n_k\}\rangle \equiv |n_1, n_2, \ldots, n_k, \ldots\rangle, \quad \hat{n}_k|\{n_k\}\rangle = n_k|\{n_k\}\rangle$$

Non-interacting Bose particles

- Hamiltonian

$$\hat{H}_0 = -\frac{\hbar^2}{2m}\int d^3x\, \hat{\Psi}^\dagger(\boldsymbol{x})\boldsymbol{\nabla}^2\hat{\Psi}(\boldsymbol{x}) + \int d^3x\, V(\boldsymbol{x})\hat{\Psi}^\dagger(\boldsymbol{x})\hat{\Psi}(\boldsymbol{x})$$

- Expansion with the solutions of the Schrödinger equation

$$-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\phi_k(\boldsymbol{x}) + V(\boldsymbol{x})\phi_k(\boldsymbol{x}) = \epsilon_k\phi_k(\boldsymbol{x})$$

$$\hat{H}_0 = \sum_k \epsilon_k \hat{a}_k^\dagger \hat{a}_k$$

- $|\{n_k\}\rangle$ is the energy eigenstate with eigenvalue $\sum_k \epsilon_k n_k$.

Statistical mechanics of non-interacting Bose particles

- Grand canonical distribution ($\beta = 1/k_B T$)

$$\hat{\rho}_{\mathrm{GC}} = \sum_{n_1, n_2, \ldots = 0}^{\infty} \frac{\exp\left[-\beta \sum_k (\epsilon_k - \mu) n_k\right]}{Z} |\{n_k\}\rangle\langle\{n_k\}|$$

- Average occupation number (Bose distribution function)

$$\langle \hat{n}_k \rangle = \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1}$$

- Chemical potential μ is determined from total particle number N as

$$\sum_k \langle \hat{n}_k \rangle = \sum_k \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1} = N.$$

BEC of non-interacting Bose particles

- Integral approximation

$$N = \sum_k \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1} \approx \langle \hat{n}_0 \rangle + \int_0^\infty d\epsilon \, \frac{D(\epsilon)}{e^{\beta(\epsilon - \mu)} - 1}$$

$$\text{Density of states} \quad D(\epsilon) \propto \begin{cases} \epsilon^{1/2} & \text{3D free space} \\ \epsilon^2 & \text{3D harmonic trap} \end{cases}$$

- The second term has finite maximum values in these cases.

  $\Rightarrow$ When $T < T_c$, almost all particles are in the ground mode.

$$\langle \hat{n}_0 \rangle \sim N$$ **Bose-Einstein condensation**

- At zero temperature, only the ground mode is occupied.

$$|N\rangle_0 \equiv |N, 0, 0, \ldots\rangle \qquad \text{number state}$$

BEC of weakly interacting Bose particles

- Hamiltonian

$$\hat{H} = \hat{H}_0 + \frac{g}{2}\int d^3x\, \hat{\Psi}^\dagger(\boldsymbol{x})\hat{\Psi}^\dagger(\boldsymbol{x})\hat{\Psi}(\boldsymbol{x})\hat{\Psi}(\boldsymbol{x})$$

$$g = \frac{4\pi\hbar^2 a}{m} \qquad a\text{: scattering length}$$

- For a dilute gas at zero temperature, the ground state is

$$|N\rangle_0 \equiv |N,0,0,\ldots\rangle \qquad \text{number state}$$

  and the ground mode is determined by Gross-Pitaevskii equation

$$-\frac{\hbar^2}{2m}\nabla^2\phi_0(\boldsymbol{x}) + V(\boldsymbol{x})\phi_0(\boldsymbol{x}) + Ng|\phi_0(\boldsymbol{x})|^2\phi_0(\boldsymbol{x}) = \mu_0\phi_0(\boldsymbol{x}).$$

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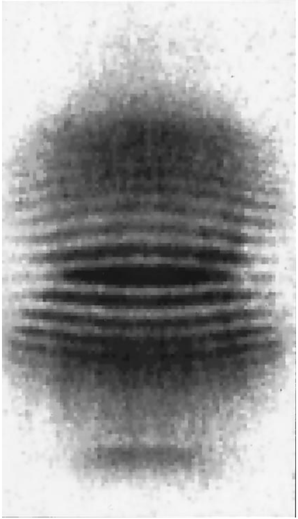
Experimental system

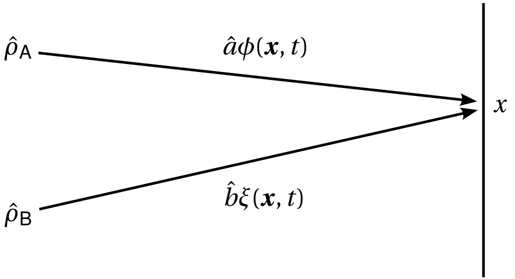


- 5×10^6 Na atoms (repulsive) trapped in a double-well potential.
- Further cooled and BEC occured.
- After 40 ms time-of-flight, observed atomic density by absorption imaging.

Andrews *et al.*, Science **275**, 637 (1997)

Single mode theory — naive calculation

- Non-interacting particles, zero temperature.



$$\hat{\Psi}(\boldsymbol{x}, t) = \hat{a}\phi(\boldsymbol{x}, t) + \hat{b}\xi(\boldsymbol{x}, t)$$

(Heisenberg picture, ϕ, ξ: solutions of Schrödinger equation)

- Average particle density $\propto$ probability to detect a particle

$$\langle \hat{\Psi}^\dagger(\boldsymbol{x}, t)\hat{\Psi}(\boldsymbol{x}, t)\rangle = \langle \hat{a}^\dagger \hat{a}\rangle|\phi|^2 + \langle \hat{b}^\dagger \hat{b}\rangle|\xi|^2 + \underline{\langle \hat{a}\rangle\langle \hat{b}^\dagger\rangle\phi\xi^* + \text{c.c.}}$$

Single mode theory — naive calculation

- Average particle density $\propto$ probability to detect a particle

$$\langle \hat{\Psi}^\dagger(\boldsymbol{x},t)\hat{\Psi}(\boldsymbol{x},t)\rangle = \langle \hat{a}^\dagger \hat{a}\rangle |\phi|^2 + \langle \hat{b}^\dagger \hat{b}\rangle |\xi|^2 + \underline{\langle \hat{a}\rangle\langle \hat{b}^\dagger\rangle \phi\xi^* + \text{c.c.}}$$

- BEC at zero temperature: number state $|N\rangle$

$$\Rightarrow \quad \langle \hat{a}\rangle = \langle N|\hat{a}|N\rangle = 0$$

$$\Rightarrow \quad \langle \hat{\Psi}^\dagger(\boldsymbol{x},t)\hat{\Psi}(\boldsymbol{x},t)\rangle = \langle \hat{a}^\dagger \hat{a}\rangle |\phi(\boldsymbol{x},t)|^2 + \langle \hat{b}^\dagger \hat{b}\rangle |\xi(\boldsymbol{x},t)|^2$$

- **No interference term!**
- $\langle \hat{a}\rangle \neq 0$ is required for the interference terms to remain.

Coherent state

- If the state is a *coherent state*,

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{N=0}^{\infty} \frac{\alpha^N}{\sqrt{N!}} |N\rangle,$$

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \quad \Rightarrow \quad \langle\hat{a}\rangle = \alpha.$$

$$\langle\hat{\Psi}^\dagger(\boldsymbol{x},t)\hat{\Psi}(\boldsymbol{x},t)\rangle = |\alpha\phi(\boldsymbol{x},t)|^2 + |\beta\xi(\boldsymbol{x},t)|^2 + \underline{\alpha\beta^*\phi(\boldsymbol{x},t)\xi^*(\boldsymbol{x},t) + \text{c.c.}}$$

- The interference terms remain.
- *However*, the coherent state contains *a superposition of different number states*. In the preparation of two *independent* BECs, there is *no* process that generates such superpositions.

Connection between number state and coherent state

- Coherent state satisfies

$$\langle \hat{a} \rangle = \alpha = |\alpha| e^{i \arg \alpha}$$

$\Rightarrow$ has a well-defined phase $\arg \alpha$.

- Number state is the superposition over the phase:

$$|N\rangle \propto \int \frac{d\varphi}{2\pi} e^{-iN\varphi} |\alpha e^{i\varphi}\rangle. \qquad \alpha\text{: arbitrary}$$

$\Rightarrow$ The phase φ is completely random. (U(1) symmetry)

Q&A

Q. What if the particles are exchanged with the environment?

A. Since the particle number in *the whole system* is conserved,

there is no superpositions.

$$c_1|N\rangle_{\mathrm{sys}}|M\rangle_{\mathrm{env}} + c_2|N-1\rangle_{\mathrm{sys}}|M+1\rangle_{\mathrm{env}}$$

$$\xrightarrow{\mathrm{Tr_{env}}} |c_1|^2|N\rangle_{\mathrm{sys}}\langle N| + |c_2|^2|N-1\rangle_{\mathrm{sys}}\langle N-1|$$

Q&A

Q. Perhaps the U(1) symmetry is spontaneously broken.

A. Even if the energy expectation value of the state without the symmetry is low, there is no realistic process to realize such a state.

"… it is neither necessary nor desirable to introduce the idea of spontaneously broken U(1) symmetry, …"

A. J. Leggett

Q&A

Q. You can superpose the particle number *difference*.

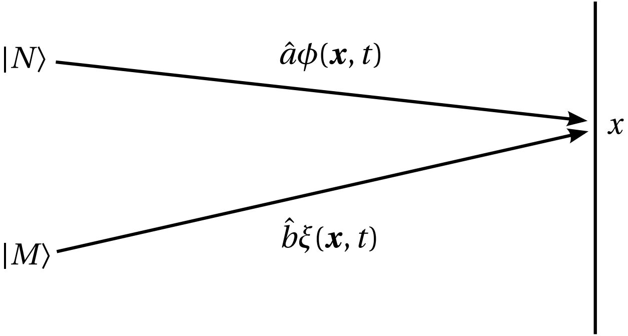
A. Yes. Phase state

$$\frac{1}{\sqrt{2^N N!}}\left(\hat{a}^\dagger + e^{i\varphi}\hat{b}^\dagger\right)^N|0\rangle = \sum_k \sqrt{\frac{N!}{k!(N-k)!}}|k\rangle_a|N-k\rangle_b$$

However, this state cannot be prepared *independently*.

In other words, it is the same situation as the “interference of single particles.”

How do two number states interferes?



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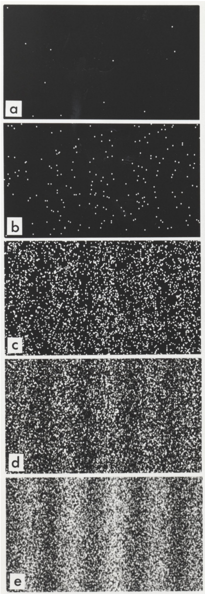
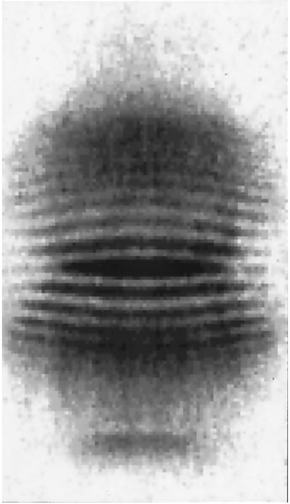


- Each particle *independently* obeys *a single distribution.*
- *Interference pattern* = *average particle density*

Interference between two independent BECs

- *A single shot* measurement for a system with a large number of particles
- *Interference pattern*

  $\neq$ *average particle density*
- Need to consider *backaction* of the measurement.



Measurements induce interference

Javanainen and Yoo, PRL **76**, 161 (1996)

- Two BECs with wave number k and $-k$

$$\hat{\Psi}(x) \propto \hat{a}e^{ikx} + \hat{b}e^{-ikx}, \quad |\psi_0\rangle = |N, N\rangle$$

- Average particle density

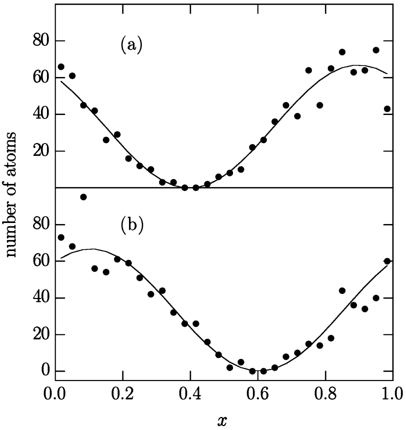
$$\langle\psi_0|\hat{\Psi}^\dagger(x)\hat{\Psi}(x)|\psi_0\rangle \propto \langle\psi_0|(\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b})|\psi_0\rangle = 2N$$

- State after a single particle detection at x_1

$$|\psi_1\rangle \propto \hat{\Psi}(x_1)|\psi_0\rangle = e^{ikx_1}|N-1, N\rangle + e^{-ikx_1}|N, N-1\rangle$$

$$\langle\psi_1|\hat{\Psi}^\dagger(x)\hat{\Psi}(x)|\psi_1\rangle \propto K_0 + \underline{K_1 \cos k(x - x_1)}$$

Measurements induce interference



Numerical simulation

Particle number $N = 1000$

(PRL **76**, 161 (1996))

(a) Number state + measurement

(b) Coherent state

Measurements localize the relative phase

Sanders *et al.*, PRA **68**, 042329 (2003)

- Number state can be expanded with coherent state

$$|N\rangle \propto \int \frac{d\varphi}{2\pi} e^{-iN\varphi} \left|\alpha e^{i\varphi}\right\rangle$$

- State after a single particle detection at x

$$\hat{\Psi}(x)|N,N\rangle \propto \int \frac{d\varphi}{2\pi} \int \frac{d\varphi'}{2\pi} C(\varphi,\varphi') \left|\alpha e^{i\varphi}\right\rangle \left|\alpha e^{i\varphi'}\right\rangle$$

$$C(\varphi,\varphi') = e^{-iN(\varphi+\varphi')} \left(e^{i(\varphi+kx)} + e^{i(\varphi'-kx)}\right)$$

$$|C(\varphi,\varphi')|^2 = 2\cos^2(\varphi-\varphi'+2kx)$$

Measurements localize the relative phase

- After m particles are detected at the same position $x = 0$

$$|C(\varphi, \varphi')|^{2m} \propto \cos^{2m}(\varphi - \varphi') \propto \exp\left[-\frac{1}{4} m(\varphi - \varphi')^2\right]$$

- Width of the distribution of the relative phase

$$\Delta(\varphi - \varphi') \sim \frac{1}{\sqrt{m}} \to 0 \quad (m \to \infty)$$

$$|C(\varphi, \varphi')|^{2m} \to \delta(\varphi - \varphi' - \pi/2) + \delta(\varphi - \varphi' + \pi/2)$$

- *The relative phase $\varphi - \varphi'$ is localized around $\pm\pi/2$.*

Interacting case

Paraoanu, PRA **77**, 041605R (2008)

- With the initial state $|N/2, N/2\rangle$, the state is

$$\int \frac{d\varphi}{2\pi} \left|\Phi_\varphi(t)\right\rangle_N, \quad \left|\Phi_\varphi(t)\right\rangle_N = \frac{1}{\sqrt{N!}} \left[\int d^3x\, \Phi_\varphi(\boldsymbol{x}, t) \hat{\Psi}^\dagger(\boldsymbol{x})\right]^N |0\rangle,$$

  $\Phi_\varphi(\boldsymbol{x}, t)$: solution of time-dependent GP equation evolved from initial wave function

$$\Phi_\varphi(\boldsymbol{x}, 0) = \frac{1}{\sqrt{2}} \left[\phi(\boldsymbol{x}) e^{-i\varphi/2} + \xi(\boldsymbol{x}) e^{i\varphi/2}\right].$$

- Similar localization effect can be derived.
- The average density *itself* shows ripples due to the interaction.

Further questions

- Measurements induce relative phase localization.
- Is there any difference in observed values in the case of the number state and that of the coherent state?
- Is the *state change by measurements* always necessary to describe interference?
- All observable values are probabilistic variables in quantum theory. We want to investigate their probabilistic behavior.

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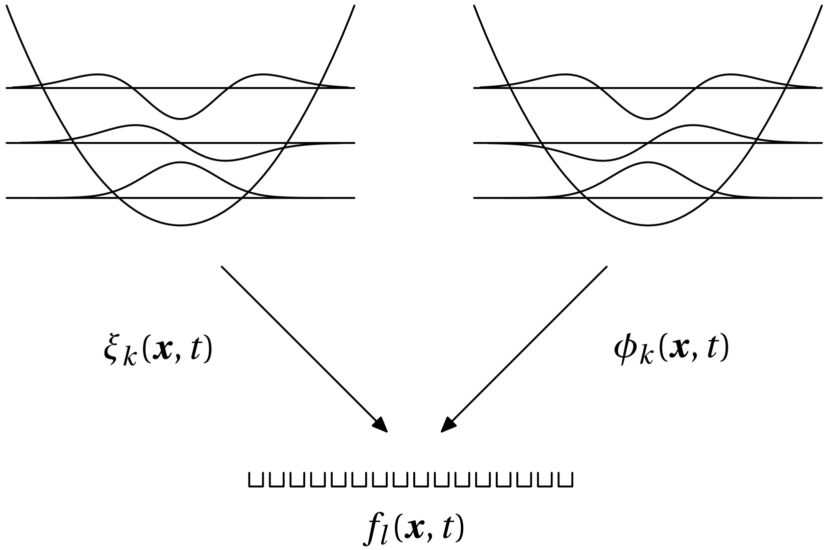
Situation



Situation

$$\hat{\Psi}(\boldsymbol{x},t) = \sum_k \left[\hat{a}_k(t)\phi_k(\boldsymbol{x},t) + \hat{b}_k(t)\xi_k(\boldsymbol{x},t)\right]$$

- Expansion with detector modes

$$\hat{\Psi}(\boldsymbol{x},t) = \sum_l \hat{d}_l(t) f_l(\boldsymbol{x},t)$$

$$\hat{d}_l(t) = \sum_k \left[p_{lk}(t)\hat{a}_k(t) + q_{lk}(t)\hat{b}_k(t)\right]$$

$$p_{lk}(t) = \int d^3x\, f_l^*(\boldsymbol{x},t)\phi_k(\boldsymbol{x},t), \quad q_{lk}(t) = \int d^3x\, f_l^*(\boldsymbol{x},t)\xi_k(\boldsymbol{x},t)$$

- Observed particle number

$$\hat{n}_l(t) = \hat{d}_l^\dagger(t)\hat{d}_l(t)$$

Single mode approximation

$$\hat{\Psi}(\boldsymbol{x}, t) = \hat{a}(t)\phi(\boldsymbol{x}, t) + \hat{b}(t)\xi(\boldsymbol{x}, t)$$

- Expansion with detector modes

$$\hat{\Psi}(\boldsymbol{x}, t) = \sum_l \hat{d}_l(t) f_l(\boldsymbol{x}, t)$$

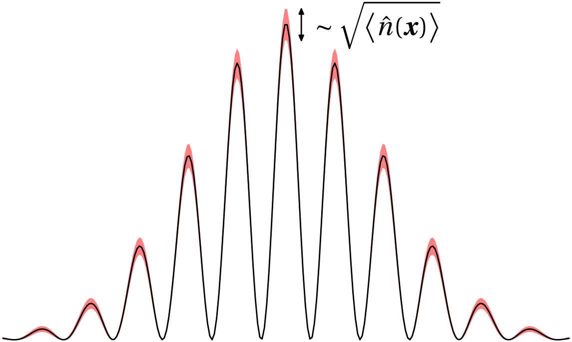
$$\hat{d}_l(t) = p_l(t)\hat{a}(t) + q_l(t)\hat{b}(t)$$

$$p_l(t) = \int d^3x\, f_l^*(\boldsymbol{x}, t)\phi(\boldsymbol{x}, t), \quad q_l(t) = \int d^3x\, f_l^*(\boldsymbol{x}, t)\xi(\boldsymbol{x}, t)$$

- Observed particle number

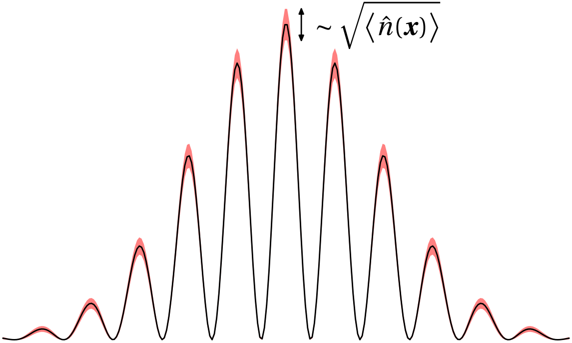
$$\hat{n}_l(t) = \hat{d}_l^\dagger(t)\hat{d}_l(t) = \left[p_l^*(t)\hat{a}^\dagger(t) + q_l^*(t)\hat{b}^\dagger(t)\right]\left[p_l(t)\hat{a}(t) + q_l(t)\hat{b}(t)\right]$$

Why coherent states are compatible with interference



- Coherent state $|\sqrt{N} e^{i\varphi_a}\rangle |\sqrt{N} e^{i\varphi_b}\rangle$
- Expectation value $\langle \hat{n}_l \rangle = N\left(|p_l|^2 + |q_l|^2 + 2\,\mathrm{Re}\, p_l q_l^* e^{i(\varphi_a - \varphi_b)}\right)$
- Variance $\sqrt{\left\langle \left(\hat{n}_l - \langle \hat{n}_l \rangle\right)^2 \right\rangle} = \sqrt{2\langle \hat{n}_l \rangle} \ll \langle \hat{n}_l \rangle$ **small**

Why coherent states are compatible with interference

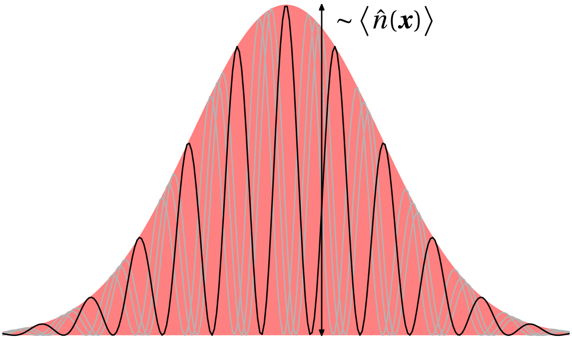


- Single-shot measured value has almost definite value.

$$n_l = N\left[|p_l|^2 + |q_l|^2 + 2|p_l||q_l|\cos(\varphi_a - \varphi_b + \varphi)\right] + \delta_l,$$

$$\varphi = \arg p_l q_l^*, \quad \langle \delta_l \rangle = 0, \quad \sqrt{\langle \delta_l^2 \rangle} \sim \sqrt{\langle \hat{n}_l \rangle} \ll \langle \hat{n}_l \rangle$$

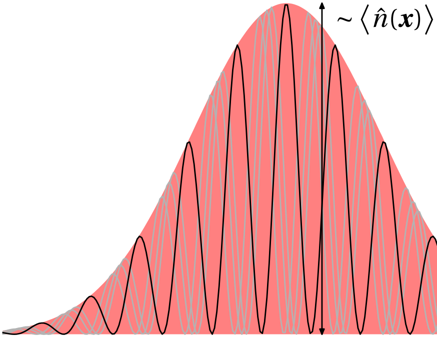
Number state



- Number state $|N\rangle|N\rangle$
- Expectation value $\langle \hat{n}_l \rangle = \left(|p_l|^2 + |q_l|^2\right) N$
- Variance $\sqrt{\left\langle \left(\hat{n}_l - \langle \hat{n}_l \rangle\right)^2 \right\rangle} = \sqrt{2|p_l|^2|q_l|^2 N^2 + O(N)} \sim \langle \hat{n}_l \rangle$

large

Number state



Even if a single-shot measurement shows interference, *the phase* of the interference is random to experiment to experiment.

⇒ The expectation values do *not* show interference and the variance is *very large*.

What should we compare?

- Coherent state: Single-shot measurement shows interference and the phase is definite.

$$n_l = N\left[|p_l|^2 + |q_l|^2 + 2|p_l||q_l|\cos(\varphi_a - \varphi_b + \varphi)\right] + \delta_l$$

$$\varphi = \arg p_l q_l^*, \quad \langle \delta_l \rangle = 0, \quad \sqrt{\langle \delta_l^2 \rangle} \sim \sqrt{\langle \hat{n}_l \rangle} \ll \langle \hat{n}_l \rangle$$

- *Relative phase randomized coherent state*

$$\hat{\rho}_{\mathrm{PRC}} = \int \frac{d\varphi'}{2\pi} |\sqrt{N} e^{i\varphi'}\rangle\langle\sqrt{N} e^{i\varphi'}| \otimes |\sqrt{N} e^{-i\varphi'}\rangle\langle\sqrt{N} e^{-i\varphi'}|$$

$\Rightarrow$ φ in the above equation becomes a completely random value $\Rightarrow$ Single-shot *shows* interference, expectation does *not*.

What should we compare?

- Compare the probability distribution of observed particle numbers $\hat{n}_l$ in the following two case:
- *Number state* (or mixture of them)

$$|N\rangle|M\rangle \quad \left(\sum_N p_N |N\rangle\langle N| \otimes \sum_M p'_M |M\rangle\langle M|\right)$$

- *Relative phase randomized coherent state*

$$\hat{\rho}_{\mathrm{PRC}} = \int \frac{d\varphi'}{2\pi} |\alpha e^{i\varphi'}\rangle\langle\alpha e^{i\varphi'}| \otimes |\beta e^{-i\varphi'}\rangle\langle\beta e^{-i\varphi'}|$$

Correlation functions

- Want to calculate correlation functions of measured value $\hat{n}_l$:

$$\left\langle \hat{n}_1^{t_1} \cdots \hat{n}_M^{t_M} \right\rangle, \qquad \hat{n}_l = \left[p_l^* \hat{a}^\dagger + q_l^* \hat{b}^\dagger\right]\left[p_l \hat{a} + q_l \hat{b}\right].$$

- Calculation of normal-ordered product average

$$\left\langle \left(\hat{a}^\dagger\right)^{s'} \hat{a}^s \left(\hat{b}^\dagger\right)^{t-s} \hat{b}^{t-s'} \right\rangle \prod_l \left(p_l^*\right)^{s_l'} p_l^{s_l} \left(q_l^*\right)^{t_l - s_l} q_l^{t_l - s_l'},$$

$$t = \sum_l t_l, \quad s = \sum_l s_l, \quad s' = \sum_l s_l'$$

is sufficient.

Evaluation of correlation functions

- *Number state* (or mixture of them)

$$\left\langle \left(\hat{a}^{\dagger}\right)^{s'} \hat{a}^{s} \left(\hat{b}^{\dagger}\right)^{t-s} \hat{b}^{t-s'} \right\rangle = \left\langle \left(\hat{a}^{\dagger}\right)^{s} \hat{a}^{s} \right\rangle \left\langle \left(\hat{b}^{\dagger}\right)^{t-s} \hat{b}^{t-s} \right\rangle \delta_{ss'}$$

- *Relative phase randomized coherent state*

$$\left\langle \left(\hat{a}^{\dagger}\right)^{s'} \hat{a}^{s} \left(\hat{b}^{\dagger}\right)^{t-s} \hat{b}^{t-s'} \right\rangle_{\mathrm{PRC}} = \left(\alpha^{*}\right)^{s} \alpha^{s} \left(\beta^{*}\right)^{t-s} \beta^{t-s} \delta_{ss'}$$

- Compare the ordinary *coherence function* $\langle \hat{G}_s \rangle = \left\langle \left(\hat{a}^{\dagger}\right)^{s} \hat{a}^{s} \right\rangle$ and $\left(\alpha^{*}\right)^{s} \alpha^{s}$.
- Not *mutual* but *self* coherence is important.

Number state

- For a number state $|N\rangle$,

$$\langle \hat{G}_s \rangle = \frac{N!}{(N-s)!}$$

- With $\alpha = \sqrt{N}$, $\left(\alpha^*\right)^s \alpha^s = N^s$. Therefore

$$\frac{\langle \hat{G}_s \rangle - N^s}{N^s} \sim \frac{1}{N}. \qquad (s \ll N)$$

- The difference between these two states is $O(1/N)$.
- With sufficiently large N, these two states are practically equivalent.

Poissonian distribution

- Laser cavity

$$\hat{\rho} = e^{-\overline{N}} \sum_{N=0}^{\infty} \frac{\overline{N}^N}{N!} |N\rangle\langle N| = \int \frac{d\varphi}{2\pi} \left| \sqrt{\overline{N}} e^{i\varphi} \right\rangle \left\langle \sqrt{\overline{N}} e^{i\varphi} \right|$$

- With $\alpha = \sqrt{\overline{N}}$, $\left(\alpha^*\right)^s \alpha^s = \overline{N}^s$. In this case,

$$\langle \hat{G}_s \rangle = \overline{N}^s.$$

- $\hat{\rho}$ and the phase randomized coherent state is completely equivalent.

Grand canonical distribution

- Thermal state without condensation

$$\hat{\rho} = \frac{1}{\overline{N}+1} \sum_{N=0}^{\infty} \left(\frac{\overline{N}}{\overline{N}+1} \right)^{N} |N\rangle\langle N|$$

$$\langle \hat{G}_s \rangle = s! \overline{N}^{s}$$

- With $\alpha = \sqrt{\overline{N}}$,

$$\frac{\langle \hat{G}_s \rangle - \overline{N}^{s}}{\overline{N}^{s}} = s! - 1 \sim 1$$

- Thermal gas is *not coherent.*

Finite temperature (multimode) case

$$\hat{n}_l = \sum_{k,k'} \left(p^*_{lk'} p_{lk} \hat{a}^\dagger_{k'} \hat{a}_k + q^*_{lk'} q_{lk} \hat{b}^\dagger_{k'} \hat{b}_k + p^*_{lk'} q_{lk} \hat{a}^\dagger_{k'} \hat{b}_k + q^*_{lk'} p_{lk} \hat{b}^\dagger_{k'} \hat{a}_k \right)$$

- *Without condensation*

  Grand canonical distribution ⇒ populations are statistically independent ⇒ correlation function factorize ⇒ single mode theory with grand canonical distribution ⇒ no interference.

- *With condensation*

  Typical matrix element for $\hat{a}_k$ $(k \neq 0)$ is small and negligible ⇒ single mode theory with canonical distribution

BEC at finite temperature

- BEC at finite temperature obeys the canonical distribution.
- Grand canonical average (N, $V \to \infty$ with $\rho = N/V$)

$$\frac{1}{V^s}\left\langle \frac{N!}{(N-s)!} \right\rangle_{\mathrm{GC}} = s!\left(\frac{\langle N\rangle_{\mathrm{GC}}}{V}\right)^s \to s!(\rho - \rho_c)^s$$

- Canonical average is calculated from grand canonical average[1]

$$\frac{1}{V^s}\left\langle \frac{N!}{(N-s)!} \right\rangle_{\mathrm{C}} = \frac{1}{s!}\frac{1}{V^s}\left\langle \frac{N!}{(N-s)!} \right\rangle_{\mathrm{GC}} \to (\rho - \rho_c)^s$$

- This exhibits Poissonian-like distribution of the condensate.

[1] c.f. Ziff *et al.*, Phys. Rep. **32**, 169 (1977).

Summary and outlook

- We can intuitively understand interference between two independent BECs with relative phase localization by measurements.
- Alternatively, we can use correlation function analysis to quantitatively evaluate the interference, including finite teperature case.
- How about interacting case? Can we separate the effect of *initial correlation* and that of *interaction*?